Journal of Statistical Physics, Vol. 79, Nos. 3/4, 1995

A Remark on the Hamiltonian Formalism for Incompressible Flows

Dario Benedetto[1]

Received November 20, 1994

We revisit the Hamiltonian formalism for incompressible flows as introduced by Oseledets. Our aim is to clarify some ambiguities in this formalism due to the nonintegrable singularity of the kernel which defines the Hamiltonian.

KEY WORDS: Incompressible fluids; Euler equation.

Oseledets[5] introduced a Hamiltonian formalism for incompressible fluids. In this formalism the velocity field is the projection on the divergence-free fields of a new unknown, a vector field which is physically unobservable; this new variable and the flow associated with the velocity field satisfy the Hamilton equations for a Hamiltonian which is practically the energy of the fluid; i.e., the L^2-norm of the velocity field. Unfortunately, due to a nonintegrable singularity of the kernel defining the Hamiltonian in terms of the new variable, some of the conclusions in ref. 5 (including the expression of the Hamiltonian) are not correct, or at least ambiguous. Thus, even though the general strategy is interesting and fruitful, a more careful analysis is necessary. The aim of this note is to fill this gap.

Considering the term $(\mathbf{\Phi}, \gamma, \mathbf{u})$, where $\mathbf{\Phi}(\cdot, t)$: $\mathbb{R}^3 \to \mathbb{R}^3$ is a family of invertible mappings parametrized by the time $t \in \mathbb{R}$, $\gamma = \gamma(\mathbf{x}, t)$ is a time-dependent vector field, and $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the projection of $\mathbf{M}(\mathbf{x}, t) = \gamma(\mathbf{\Phi}^{-1}(\mathbf{x}, t), t)$ on the divergence-free field. Finally, $\mathbf{\Phi}$ induces the integral lines of $\mathbf{u}$. Namely

$$\gamma(\mathbf{x}, t) = \mathbf{u}(\mathbf{\Phi}(\mathbf{x}, t), t) + \nabla\psi(\mathbf{\Phi}(\mathbf{x}, t), t) \qquad (1)$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0 \qquad (2)$$

[1] Dipartimento di Matematica Pura ed Applicata, Università dell' Aquila, 67100 L'Aquila, Italy. E-mail: BENEDETTO@VXSCAQ.AQUILA.INFN.IT.

0022-4715/95/0500-0743$07.50/0

$$\begin{aligned}&\partial_t \mathbf{\Phi}(\mathbf{x}, t) = \mathbf{u}(\mathbf{\Phi}(\mathbf{x}, t), t)\\&\mathbf{\Phi}(\mathbf{x}, 0) = \mathbf{x}\end{aligned} \tag{3}$$

Moreover, we assume

$$\partial_t \gamma(\mathbf{x}, t) = -(\nabla \mathbf{u})^t (\mathbf{\Phi}(\mathbf{x}, t), t)\, \gamma(\mathbf{x}) \tag{4}$$

where the matrix $(\nabla \mathbf{u})^t$ is $(\nabla \mathbf{u})^t_{i,j} = \partial u_j / \partial x_i$. The system (3), (4) with the constraints (1), (2) is equivalent to the Euler equation in the following sense.

Proposition 1. If Eqs. (1)–(4) hold, then $\mathbf{u}$ satisfies

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla P \tag{5}$$

for some scalar field P (the pressure), which satisfies

$$P = \partial_t \psi + (\mathbf{u} \cdot \nabla)\, \psi + \frac{\mathbf{u}^2}{2} \tag{6}$$

Conversely, if $\mathbf{u}$ is a divergence-free field which satisfies the Euler equation (5), and ψ is a scalar field which satisfies Eq. (4), then $\gamma = \mathbf{u}(\mathbf{\Phi}) + \nabla \psi(\mathbf{\Phi})$ satisfies Eq. (2), provided that $\mathbf{\Phi}$ is the flow associated with the velocity field $\mathbf{u}$.

The proof follows formally by a calculation[1,2,4,5]; it can be made rigorous, locally in time, by using known results on the local existence of classical solutions to the Euler problem (6) (see, e.g., ref. 3).

The interest in the system (3), (4) is in its Hamiltonian structure, which we are going to exploit. It is worth mentioning that such a Hamiltonian formalism suggests a Hamiltonian particle approximation recently implemented numerically.[2,1]

Following ref. 5, we choose the canonical conjugate variables as $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x})$ and $\gamma = \gamma(\mathbf{x})$, for which Eqs. (3), (4) will be the Hamiltonian equations. Then we express the energy in terms of $\mathbf{\Phi}$ and γ.

Let $\mathscr{P}$ be the projection operator on the divergence-free field:

$$\mathscr{P}\mathbf{v} = \mathbf{v} - \nabla \Delta^{-1} \nabla \cdot \mathbf{v} \tag{7}$$

for a given smooth field $\mathbf{v} = \mathbf{v}(\mathbf{x})$. The energy of the fluid (purely kinetic) is defined by

$$E = \tfrac{1}{2} \int d\mathbf{x}\, \mathbf{u}^2 = \tfrac{1}{2} \int d\mathbf{x}\, \gamma(\mathbf{\Phi}^{-1}) \cdot \mathscr{P}(\gamma(\mathbf{\Phi}^{-1})) \tag{8}$$

It seems reasonable to choose the right-hand side of (8) as the Hamiltonian, but if we do this, we do not obtain the system (3), (4). The right choice is the following. Let us denote

$$\boldsymbol{\mu}(\boldsymbol{\gamma}, \boldsymbol{\Phi})(\mathbf{x}) = \boldsymbol{\gamma}(\boldsymbol{\Phi}^{-1}(\mathbf{x})) \det \nabla(\boldsymbol{\Phi}^{-1}(\mathbf{x})) \tag{9}$$

Then the Hamiltonian which we propose is

$$H = \frac{1}{2}\int \boldsymbol{\mu}\cdot\mathscr{P}\boldsymbol{\mu} \tag{10}$$

Remark. If $\boldsymbol{\Phi}$ is an incompressible mapping, $H = E$, but we need expression (10) for the Hamiltonian to do the correct variation on $\boldsymbol{\Phi}$ without the incompressibility constraint. The incompressibility will be just a consequence of the Hamiltonian equations.

We perform the variations of such H observing that

$$\begin{aligned} &H(\boldsymbol{\gamma}+\delta\boldsymbol{\gamma}, \boldsymbol{\Phi}) - H(\boldsymbol{\gamma}, \boldsymbol{\Phi}) \\ &\quad = \int d\mathbf{x}(\boldsymbol{\mu}(\boldsymbol{\gamma}+\delta\boldsymbol{\gamma}, \boldsymbol{\Phi})(\mathbf{x}) - \boldsymbol{\mu}(\boldsymbol{\gamma}, \boldsymbol{\Phi})(\mathbf{x}))\cdot\mathscr{P}\boldsymbol{\mu}(\mathbf{x}) + o(\boldsymbol{\delta\gamma}) \\ &\quad = \int d\mathbf{y}\, \delta\boldsymbol{\gamma}(\mathbf{y})\cdot(\mathscr{P}\boldsymbol{\mu})(\boldsymbol{\Phi}(\mathbf{y})) + o(\delta\boldsymbol{\gamma}) \end{aligned} \tag{11}$$

$$\begin{aligned} &H(\boldsymbol{\gamma}, \boldsymbol{\Phi}+\delta\boldsymbol{\Phi}) - H(\boldsymbol{\gamma}, \boldsymbol{\Phi}) \\ &\quad = \int d\mathbf{x}(\boldsymbol{\mu}(\boldsymbol{\gamma}, \boldsymbol{\Phi}+\delta\boldsymbol{\Phi})(\mathbf{x}) - \boldsymbol{\mu}(\boldsymbol{\gamma}, \boldsymbol{\Phi})(\mathbf{x}))\cdot(\mathscr{P}\boldsymbol{\mu})(\mathbf{x}) + o(\boldsymbol{\delta\Phi}) \\ &\quad = \int d\mathbf{y}\, \boldsymbol{\gamma}(\mathbf{y})\cdot((\mathscr{P}\boldsymbol{\mu})(\boldsymbol{\Phi}+\delta\boldsymbol{\Phi})(\mathbf{y}) - (\mathscr{P}\boldsymbol{\mu})(\boldsymbol{\Phi}(\mathbf{y}))) + o(\boldsymbol{\delta\Phi}) \\ &\quad = \int d\mathbf{y}(\nabla\mathscr{P}\boldsymbol{\mu})'\,(\boldsymbol{\Phi}(\mathbf{y}))\,\boldsymbol{\gamma}(\mathbf{y})\cdot\delta\boldsymbol{\Phi}(\mathbf{y}) + o(\boldsymbol{\delta\Phi}) \end{aligned} \tag{12}$$

[in (12) and (13) we have used the changes of variables $\mathbf{x} = \boldsymbol{\Phi}(\mathbf{y})$ and $\mathbf{x} = (\boldsymbol{\Phi}+\delta\boldsymbol{\Phi})(\mathbf{y})$].

Then the Hamiltonian equations are

$$\begin{aligned} \partial_t\boldsymbol{\Phi} &= (\mathscr{P}\boldsymbol{\mu})(\boldsymbol{\Phi}) \\ \partial_t\boldsymbol{\gamma} &= -(\nabla\mathscr{P}\boldsymbol{\mu})'\,(\boldsymbol{\Phi})\,\boldsymbol{\gamma} \end{aligned} \tag{13}$$

We note that $\mathscr{P}\boldsymbol{\mu}$ is a divergence-free field and that $\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{x}$ at time $t = 0$. Then from the first equation of (13) it follows that $\det \nabla\boldsymbol{\Phi} = 1$. Consequently,

$$\mathscr{P}\boldsymbol{\mu} = \mathscr{P}(\boldsymbol{\gamma}(\boldsymbol{\Phi}^{-1})) = \mathbf{u} \tag{14}$$

Equations (13) with the condition (14) are exactly the system (3), (4).

Remark. We can choose other laws of dependence of the Hamiltonian on $\det \nabla\boldsymbol{\Phi}$ (unobservable, ...), which give Hamilton equations, different from (3), (4), equivalent to the Euler equation (see also ref. 1), but not so interesting for numerical implementations. For example, if we choose the right side of (8) as Hamiltonian, we obtain the following Hamilton equations

$$\begin{aligned} &\partial_t \boldsymbol{\Phi} = (\mathscr{P}\boldsymbol{\mu})(\boldsymbol{\Phi}) \det \nabla\boldsymbol{\Phi} \\ &\partial_t \boldsymbol{\gamma} = -((\nabla \mathscr{P}\boldsymbol{\mu})'(\boldsymbol{\Phi})\,\boldsymbol{\gamma} - \nabla(\mathscr{P}\boldsymbol{\mu}\cdot\gamma(\boldsymbol{\Phi}^{-1}) \det \nabla(\boldsymbol{\Phi}^{-1}))(\boldsymbol{\Phi})) \det \nabla\boldsymbol{\Phi} \end{aligned} \tag{15}$$

From the first of equations (15) we obtain

$$\partial_t \det \nabla\boldsymbol{\Phi} = \det \nabla\boldsymbol{\Phi} \operatorname{div}(\mathscr{P}\boldsymbol{\mu} \det \nabla(\boldsymbol{\Phi}^{-1}))(\boldsymbol{\Phi}) \tag{16}$$

which is identically solved by $\det \nabla\boldsymbol{\Phi} = 1$ since $\operatorname{div}(\mathscr{P}\boldsymbol{\mu}) = 0$. Then system (15) is equivalent to

$$\begin{aligned} &\partial_t \boldsymbol{\Phi} = \mathbf{u}(\boldsymbol{\Phi}) \\ &\partial_t \boldsymbol{\gamma} = -(\nabla\mathbf{u})'(\boldsymbol{\Phi})\,\gamma + \nabla(\mathbf{u}\cdot\boldsymbol{\gamma}(\boldsymbol{\Phi}^{-1}))(\boldsymbol{\Phi}) \end{aligned} \tag{17}$$

where $\mathbf{u} = \mathscr{P}(\gamma(\boldsymbol{\Phi}^{-1}))$. It is not difficult to see that also Eqs. (17) are equivalent to the Euler equation.

Now we give an explicit expression for the Hamiltonian. Since

$$\begin{aligned} &(\nabla\Delta^{-1}\nabla\cdot\mathbf{v})_i \\ &= -\frac{1}{4\pi}\int \frac{z_j}{|\mathbf{z}|^3}\,\partial_i v_j(\mathbf{x}+\mathbf{z})\,d\mathbf{z} \\ &= \lim_{\rho\to 0}\left(\frac{1}{4\pi}\int_{|\mathbf{z}|=\rho} \frac{z_i z_j}{|\mathbf{z}|^4}\,v_j(\mathbf{x}+\mathbf{z})\,d\sigma \right. \\ &\quad \left. +\frac{1}{4\pi}\int_{|\mathbf{z}|>\rho}\left(\frac{\delta_{ij}}{|\mathbf{z}|^3} - 3\frac{z_i z_j}{|\mathbf{z}|^5}\right) v_j(\mathbf{x}+\mathbf{z})\,d\mathbf{z}\right) \\ &= \frac{1}{3}\,\delta_{ij} v_j(\mathbf{x}) + \frac{1}{4\pi}\lim_{\rho\to 0}\int_{|\mathbf{z}|>\rho}\left(\frac{\delta_{ij}}{|\mathbf{z}|^3} - 3\frac{z_i z_j}{|\mathbf{z}|^5}\right) v_j(\mathbf{x}+\mathbf{z})\,d\mathbf{z} \end{aligned} \tag{18}$$

we obtain

$$(\mathscr{P}\mathbf{v})(\mathbf{x}) = \tfrac{2}{3}\mathbf{v}(\mathbf{x}) + ⨍ dy\, \mathbf{G}(\mathbf{x}-\mathbf{y})\, \mathbf{v}(\mathbf{y}) \tag{19}$$

where

$$\mathbf{G}(\mathbf{z}) = -\frac{1}{4\pi}\left(\frac{\mathbf{I}}{|\mathbf{z}|^3} - 3\,\frac{\mathbf{z}\otimes\mathbf{z}}{|\mathbf{z}|^5}\right) \tag{20}$$

In Eq. (19), $⨍$ is the Cauchy principal value. Substituting (19) in (10), we obtain

$$H = \frac{1}{3}\int d\mathbf{x}\, \frac{\gamma(\mathbf{x})^2}{\det \nabla\Phi(\mathbf{x})} + \frac{1}{2}⨍ d\mathbf{x}\, d\mathbf{y}\, \gamma(\mathbf{x})\cdot \mathbf{G}(\Phi(\mathbf{x}) - \Phi(\mathbf{y}))\, \gamma(\mathbf{y}) \tag{21}$$

where the principal value in the second term must be intended as a limit on the region $\{|\Phi(\mathbf{x}) - \Phi(\mathbf{y})| > \varepsilon\}$. We remark that in the Hamiltonian in ref. 5 there is not the diagonal term $\frac{1}{3}\int \gamma^2$.

ACKNOWLEDGMENTS

The author thanks C. Marchioro and M. Pulvirenti for helpful discussions.

REFERENCES

1. D. Benedetto, Convergence of velicity blobs method for the Euler equation, preprint (1994).
2. T. F. Buttke, Lagrangian numerical methods which preserve the Hamiltonian structure of incompressible fluid flow, *Commun. Pure Appl. Math.* (1993).
3. T. Kato, Non-stationary flows on viscous and ideal fluids in $\mathbb{R}^3$, *J. Funct. Anal.* **9**:296–309 (1972).
4. J. H. Moddocks and R. L. Pego, An unconstrained Hamiltonian formulation for incompressible fluid flow (1993), to appear.
5. V. I. Oseledets, On new way of writing the Navier–Stokes equation. The Hamiltonian formalism, *Commun. Moscow Math. Soc.* (1988) [transl., *Russ. Math. Surv.* **44** (1989)].

Communicated by R. Esposito